On countably saturated linear orders

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General definitions

- A linear order is compact, if it's compact in the order topology. This means, it is Dedekind complete, and has both endpoints.
- A linear order is linearly ordered continuum, if it is compact and connected in the order topology. This means, it is compact and dense.
- $I = [-1, 1]$.

Definition

We'll say that a linear order $(L, \leq)$ *is countably saturated, if for any countable linear orders* a*,* b*, and increasing functions* $i : a \to b$*,* $f : a \to L$*, there exists* $\tilde{f} : b \to L$*, such that* $\tilde{f} \circ i = f$*.*

Definition

We'll say that a linear order $(L, \leq)$ is countably saturated, if for any countable linear orders a, b, and increasing functions $i : a \to b$, $f : a \to L$, there exists $\tilde{f} : b \to L$, such that $\tilde{f} \circ i = f$.

There exists an equivalent definition.

Lemma

Linear order is countably saturated if and only if

- *it is dense, without endpoints,*
- *no countable increasing sequence has supremum,*
- *no countable decreasing sequence has infimum,*
- *there are no (ω, ω)-gaps: for any two sequences $\{x_n\}_{n<\omega}$, $\{y_n\}_{n<\omega}$ such that $\forall n < \omega\ x_n < x_{n+1} < y_{n+1} < y_n$, there exists z s.t. $\forall n < \omega\ x_n < z < y_n$.*

Proposition

Any countably saturated linear order contains an isomorphic copy of the real line.

Proof.

Let $(L, \leq)$ be a countably saturated linear order. It is dense, so there exists an injection $i : \mathbb{Q} \hookrightarrow L$. For any real number r, we want to define $i(r)$.
Notice that sets $i[\{q \in \mathbb{Q} : q > r\}] > i[\{q \in \mathbb{Q} : q < r\}]$ are countable. Therefore, there exists $l \in L$ such that

$$i[\{q \in \mathbb{Q} : q > r\}] > l > i[\{q \in \mathbb{Q} : q < r\}].$$

We define $i(r) = l$. □

Theorem (Hausdorff)

Assume $(L, \leq_L)$ is countably saturated, and $(X, \leq_X)$ doesn't contain a copy of ω_1 or ω_1^*. Then exists an embedding $i : X \hookrightarrow L$.

Examples

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Example (Sierpiński)

Let $Q = \{x \in \{0,1\}^{\omega_1} \,|\, \exists_{\alpha<\omega_1} x(\alpha) = 1,\ \forall \beta > \alpha\ x(\beta) = 0\}$, *with lexicographic order. This order is prime countably saturated.*

Examples

Definition

$$\mathbb{L}^{\omega_1} = \{x \in I^{\omega_1} \,|\, |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| \leq \omega\},$$

with lexicographic order. If D is compact linear order, and $d_0 \in D$ is neither least, nor greatest element of D, then we define

$$\mathbb{L}^{\omega_1}_{(D,d_0)} = \{x \in D^{\omega_1} \,|\, |\{\alpha < \omega_1 : x(\alpha) \neq d_0\}| \leq \omega\}.$$

Examples

Theorem

$\mathbb{L}^{\omega_1}$ and $\mathbb{L}^{\omega_1}_{(D,d_0)}$ are countably saturated.

Theorem

$\mathbb{L}^{\omega_1}$ is prime countably saturated. Moreover, if D is separable, compact, and $d_0 \in D$ is neither the least, nor the greatest element, $\mathbb{L}^{\omega_1}_{(D,d_0)}$ is prime.

Classification

Theorem (folklore)

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In fact, the category of countable linear orders with embeddings, has unique ω_1-Fraïssé limit.

Classification

Theorem (foklore)

Without CH, no.

Classification

Proof.

$$\mathbb{L}^{\omega_1} = \{x \in I^{\omega_1} \mid |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| \leq \omega\},$$

and

$$\{x \in I^{\omega_2} \mid |\{\alpha < \omega_2 : x(\alpha) \neq 0\}| \leq \omega\},$$

are both countably saturated. But the second contains a copy of ω_2, while the first doesn't. □

But what if we want same better examples?

Example

In the Cohen model there exists two non-isomorphic countably saturated linear orders of cardinality $\mathfrak{c}$, none of which contains copy of ω_2 or ω_2^*.

Outline of the proof:
Let M be a model of CH, $M[G]$ be extension by $Fn_{<\omega}(\omega_2)$.

- First example will be $\mathbb{L}^{\omega_1}$ (in $M[G]$). We show, that it doesn't contain copy of any linear order of cardinality ω_2, which was in M.
- For second example, we take 2^{ω_1}, and inductively define an increasing sequence of linear orders $\{R_\alpha\}_{\alpha\leq\omega_1}$, such that $R_0 = 2^{\omega_1}$, and R_{ω_1} is countably saturated.

$$(2^{\omega_1})^M \subset 2^{\omega_1} \subseteq R_{\omega_1},$$

so these two cannot be isomorphic.

Linear dimension

We'll use notion of dimension for better classification of linear orders.

Definition (V. Novák, 1963)

Let L and X be linear orders. We define dimension of X with respect to L as:

$$\text{L-dim}\, X = \min\{\alpha \in ON \mid X \hookrightarrow L^{\alpha}\}.$$

Linear dimension

Let us write down some easy observations.

Proposition

For any linear orders L, L_1, L_2, X, the following holds.

- *If* $X_1 \hookrightarrow X_2$, *then* $\text{L-dim}\, X_1 \leq \text{L-dim}\, X_2$.
- *If* $L_1 \hookrightarrow L_2$, *then* $\text{L}_1\text{-dim}\, X \geq \text{L}_2\text{-dim}\, X$.
- *If* $L_1 \hookrightarrow L_2$ *and* $L_2 \hookrightarrow L_1$, *then for every* X, $\text{L}_1\text{-dim}\, X = \text{L}_2\text{-dim}\, X$.

Linear dimension

In particular, notions of 2^ω-dim X, I-dim X, and $\mathbb{R}$-dim X coincide. We will denote them I-dim X.

Theorem (Novotný, 1953; Novák, 1963)

Let L be a linearly ordered continuum. Then for any ordinal α, L^α is a linearly ordered continuum, and $\text{L-dim}\, L^\alpha = \alpha$.

Corollary

If α is an ordinal with the property, that $\omega \cdot \alpha = \alpha$, then $\text{I-dim}\, 2^\alpha = \alpha$.

Linear dimension

Example

Assume $\mathfrak{c} = 2^{\omega_1}$. *Let* $X = I^{\omega_1}$. *Then* $\mathbb{L}^{\omega_1}_{(X,0)}$ *is a countably saturated linear order of cardinality* $\mathfrak{c}$, *without copy of* ω_2 *or* ω_2^*, *and* I-dim *equal to* ω_1^2. *In particular* $\mathbb{L}^{\omega_1}_{(X,0)}$ *is not isomorphic to* $\mathbb{L}^{\omega_1}$.

Proposition (Fleischer, 1961)

If $\text{I-dim}\, L < \omega_1$, *then L doesn't contain a copy of* ω_1 *or* ω_1^*.

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The converse doesn't hold, though the false proof was published in

I. Fleischer, *Embedding linearly ordered sets in real lexicographic products*, Fund. Math. 49 (1961)

Proposition

Let $(L, \leq)$ be countably saturated linear order. The following are equivalent:

- *L is prime.*
- $L = \bigcup_{\alpha<\omega_1} L_\alpha$, *where* $2\text{-}\dim L_\alpha < \omega_1$, *for each* $\alpha < \omega_1$.
- $L = \bigcup_{\alpha<\omega_1} L_\alpha$, *where* $\text{I-}\dim L_\alpha < \omega_1$, *for each* $\alpha < \omega_1$.

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Theorem (K., 2019)

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Theorem (K., 2019)

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Question

Can we add $\text{I-dim}\, L = \omega_1$ *to the list in previous theorem?*

Thank You for attention!

References:

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